Varianta 043

Subiectul I

a) $\sqrt{3}$. b) 3. c) $-\frac{3}{2}$. d) $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5}{13}$. e) 4. f) (2,3).

Subiectul II

1.a) 90. b) $\hat{0}$. c) $A^2 \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. d) $A_3^2 = 6$. e) r=2..

2. a) $-\frac{\pi}{2}$. b) $\frac{1}{1+x^2}$; c) $f''(x) = \frac{-2x}{(1+x^2)^2} < 0, (\forall) x \in (0,\infty)$; d) nicio solutie $\text{Im} f = (-\frac{\pi}{2}; \frac{\pi}{2})$.

$e) \int_0^1 arctgx dx = [x arctgx - \frac{1}{2}\ln(1+x^2)]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$

Subiectul III

a) $\hat{1}\cdot\hat{1} = \hat{1}; \hat{3}\cdot\hat{3} = \hat{1}, (\hat{0})^1 = \hat{0}, (\hat{2})^2 = \hat{0}.$

b) $(\hat{2}x+\hat{1})(\hat{2}x+\hat{1}) = \hat{1}; (\hat{2}x+\hat{2})^2 = \hat{0}.$

c) Fie $f \in U \cap N$. Din $f \in U \Rightarrow$ exista $g \in Z_4[X]$ astfel incat $f \cdot g = \hat{1}$.

Din $f \in N \Rightarrow$ exista $n \in \mathbf{N}, n \neq 0$ astfel incat $f^n = 0$.

Alegem n minim cu aceasta proprietate. Inmultind relatia $f \cdot g = \hat{1}$ cu f^{n-1} obtinem $\hat{0} = f^{n-1}$, ceea ce contrazice minimalitatea lui n.

d) $u^n = \hat{0}; v^m = \hat{0} \Rightarrow (u+v)^{m+n} = \sum_{k=0}^{m+n} C_{m+n}^k \cdot u^k \cdot v^{m+n-k} = \hat{0}$ deoarece $k \geq n$ sau $m+n-k \geq m$.

e) Daca $v \in N$ si $v^n = \hat{0}$ atunci si $v^{2n+1} = 0$.

Avem $u^{2n+1} = u^{2n+1} + g^{2n+1} = (u+g)(u^{2n} - u^{2n-1}g + u^{2n-2}g^2 - ... + g^{2n})$.

Daca $u \cdot g = \hat{1}$ atunci $u^{2n+1} \cdot g^{2n+1} = \hat{1} \Rightarrow (u+g)h = \hat{1}$, unde $h = (u^{2n} - u^{2n-1}g + ... + g^{2n)}) \cdot g^{2n+1}$.

f) $f \cdot g = \hat{1} \Rightarrow f(x)g(x) = \hat{1}, (\forall) x \in \mathbf{Z}_4 \Rightarrow f(\hat{0})g(\hat{0}) = \hat{1}, (\forall) x \in \mathbf{Z}_4 \Rightarrow f(\hat{0}) = g(\hat{0}) \in \{\hat{1}, \hat{3}\}$.

g) Facem inductie dupa gradul lui f. Daca grad $f = 0 \Rightarrow f = \hat{c} \overset{a}{\Rightarrow} \hat{c} \in \{\hat{0}, \hat{2}\}$.

Polinomul $h(x) = \hat{a}_{n+1} x^{n+1}$ este nilpotent($h^2 = \hat{0}$) si din d $\Rightarrow g = f - h \in N$ si conform ipotezei de inductie coeficientii lui g sunt $\hat{0}$ si $\hat{2}$.

h) Definim $v_n = \hat{2}x^n$ si $u_n = \hat{1} + v_n, n \in \mathbf{N}^*$ si avem $v_n \in N, u_n \in U, (\forall) n \in \mathbf{N}^*$.

Subiectul IV

a) $a_{n+1} - a_n = \frac{1}{2^{(n+1)!}} > 0$

$b_{n+1} - b_n = a_{n=1} - a_n + \frac{1}{(n+1)2^{(n+1)!}} - \frac{1}{n \cdot 2^{n!}} = \frac{1}{2^{(n+1)!}} + \frac{1}{(n+1)2^{(n+1)!}} =$

$= \frac{n(n+2) - (n+1)2^{nn!}}{n(n+1)2^{(n+1)!}} < 0, (\forall) n \in \mathbf{N}^*$

c) $a_n < b_n \Rightarrow a_n, b_n \in [a_1, b_1] (\forall) n \in \mathbf{N}^*$

d) Sirurile $(a_n)_n, (b_n)_n$, sunt monotone si marginite, deci convergente si

$\lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n = \lim_{n \to \infty}(b_n - a_n) = \lim_{n \to \infty} \frac{1}{n \cdot 2^{n!}} = 0.$

e) Daca prin absurd $\lim_{n \to \infty} a_n = \frac{p}{2}, p, q \in \mathbf{N}^*$ atunci $a_q < \frac{p}{q} < b_q$ sau

$\frac{1}{2^{1!}} + \frac{1}{2^{2!}} + ... + \frac{1}{2^{q!}} < \frac{p}{q} < \frac{1}{2^{1!}} + \frac{1}{2^{2!}} + ... + \frac{1}{2^{q!}} + \frac{1}{q \cdot q^{q!}}$

Inmultim cu $q \cdot q^2$ si obtinem $A < B < A + 1, A, B \in M$, relatie care nu este posibila.

f) $0 \le \lim_{n \to \infty} \frac{n^{2007}}{2^{n!}} \le \lim_{n \to \infty} \frac{n^{2007}}{2^n} = \lim_{x \to \infty} \frac{x^{2007}}{2^x} = \lim_{x \to \infty} \frac{2007 \cdot x^{2006}}{2^x \ln 2} = 0.$

g) Daca $a_n = \frac{f(n)}{g(n)}, (\forall) n \in \mathbf{N}^* \Rightarrow a_{n-1} - a_n = \frac{f(n=1)}{g(n+1)} - \frac{f(n)}{g(n)} = \frac{P(n)}{Q(n)}$, unde $P, Q \in \mathbf{R}[x]$.

Deci $P(n) = \frac{Q(n)}{2^{(n+1)!}} = 0 \Rightarrow$

$P = 0 \Rightarrow \frac{1}{2^{(n+1)!}} = 0$(contradictie).